Arithmetic-Geometric Mean

The arithmetic-geometric mean $\operatorname{agm}(a, b)$ of two numbers a and b (often also written AGM
is defined by starting with $a_0 \equiv a$ and $b_0 \equiv b$, then iterating

$$\begin{aligned} a_{n+1} &= \tfrac{1}{2}(a_n + b_n) \\ b_{n+1} &= \sqrt{a_n b_n} \end{aligned}$$

until $a_n = b_n$ to the desired precision.

a_n and b_n converge towards each other since

$$\begin{aligned} a_{n+1} - b_{n+1} &= \tfrac{1}{2}(a_n + b_n) - \sqrt{a_n b_n} \\ &= \frac{a_n - 2\sqrt{a_n b_n} + b_n}{2}. \end{aligned}$$

But $\sqrt{b_n} < \sqrt{a_n}$, so

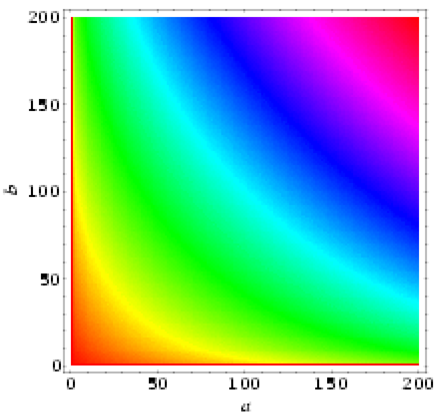
$$2 b_n < 2\sqrt{a_n b_n}.$$

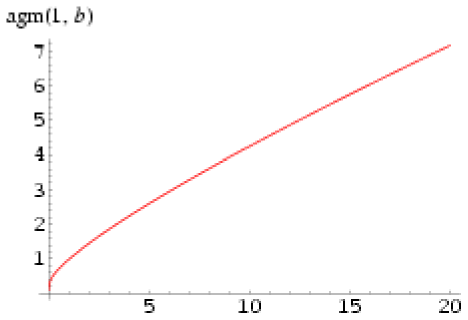
Now, add $a_n - b_n - 2\sqrt{a_n b_n}$ to each side

$$a_n + b_n - 2\sqrt{a_n b_n} < a_n - b_n,$$

so

$$a_{n+1} - b_{n+1} < \tfrac{1}{2}(a_n - b_n).$$





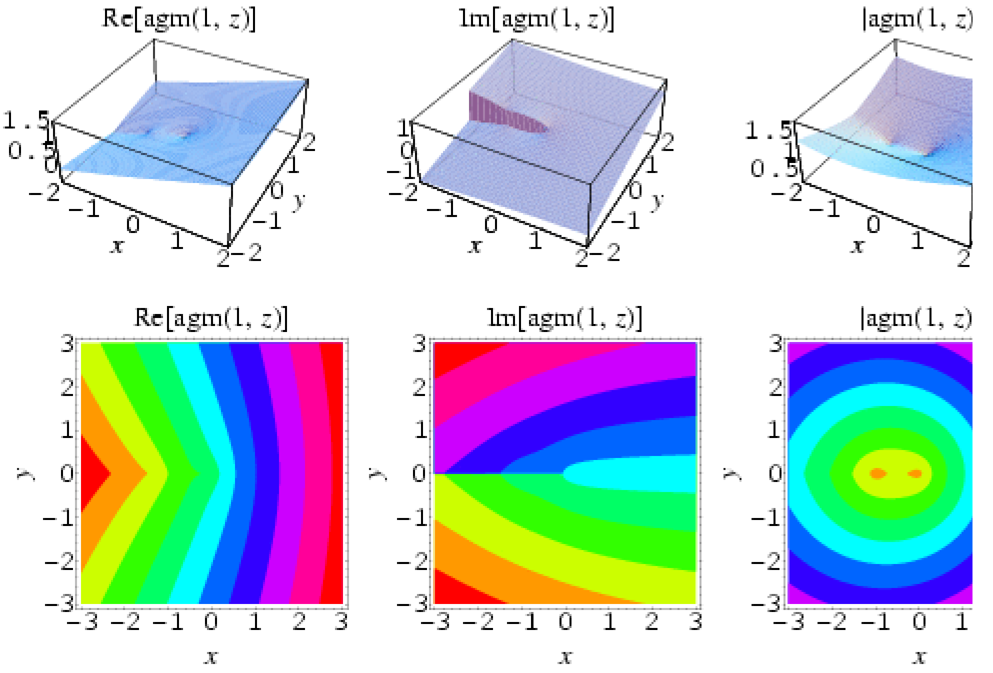
The top plots show $\operatorname{agm}(1, b)$ for $0 \le b \le 20$ and $\operatorname{agm}(a, b)$ for $0 \le a, b \le 200$, while the botto
$\operatorname{agm}(1, z)$ for complex values of z.

The AGM is very useful in computing the values of complete elliptic integrals and can also b
the inverse tangent.

It is implemented in *Mathematica* as ArithmeticGeometricMean[*a*, *b*].

$\text{agm}(a, b)$ can be expressed in closed form in terms of the complete elliptic integral of the fi

$$\text{agm}(a, b) = \frac{(a+b)\pi}{4K\left(\frac{a-b}{a+b}\right)}.$$



The definition of the arithmetic-geometric mean also holds in the complex plane, as illustrat $\text{agm}(1, z)$.

The Legendre form of the arithmetic-geometric mean is given by

$$\text{agm}(1, x) = \prod_{n=0}^{\infty} \frac{1}{2}(1 + k_n),$$

where $k_0 \equiv x$ and

$$k_{n+1} \equiv \frac{2\sqrt{k_n}}{1 + k_n}.$$

Special values of $\text{agm}(a, b)$ are summarized in the following table. The special value

$$\frac{1}{\text{agm}(1, \sqrt{2})} = 0.83462684167407318628\ldots$$

(Sloane's A014549) is called Gauss's constant. It has the closed form

$$\frac{1}{\text{agm}(1, \sqrt{2})} = \frac{2}{\pi}\int_0^1 \frac{dt}{\sqrt{1-t^4}}$$

$$= \frac{\left[\Gamma\left(\frac{1}{4}\right)\right]^2}{2\pi^{3/2}\sqrt{2}}$$

where the above integral is the lemniscate function and the equality of the arithmetic-geom integral was known to Gauss (Borwein and Bailey 2003, pp. 13-15).

$\text{agm}(a, b)$	Sloane	value

agm (1, 2)	A068521	1.4567910310469068692...
agm (1, 3)	A084895	1.8636167832448965424...
agm (1, 4)	A084896	2.2430285802876025701...
agm (1, 5)	A084897	2.6040081905309402887...

The derivative of the AGM is given by

$$\begin{aligned}\frac{\partial}{\partial b}\operatorname{agm}(a, b) &= \frac{\operatorname{agm}(a, b)}{(a-b)\,b\,\pi}\left[2\operatorname{agm}(a, b)\,E(k) - b\,\pi\right] \\ &= \frac{\pi}{8\,k\,b}\,\frac{(a+b)\,E(k) - 2\,b\,K(k)}{[K(k)]^2},\end{aligned}$$

where $k \equiv (a-b)/(a+b)$, $K(k)$ is a complete elliptic integral of the first kind, and $E(k)$ is the integral of the second kind.

A series expansion for $\operatorname{agm}(1, b)$ is given by

$$\operatorname{agm}(1, b) = -\frac{\pi}{2\ln\left(\frac{1}{4}b\right)} + \frac{\pi\left[1+\ln\left(\frac{1}{4}b\right)\right]b^2}{8\left[\ln\left(\frac{1}{4}b\right)\right]^2} + O(b^4).$$

The AGM has the properties

$$\begin{aligned}\lambda\operatorname{agm}(a, b) &= \operatorname{agm}(\lambda a, \lambda b) \\ \operatorname{agm}(a, b) &= \operatorname{agm}\left(\frac{1}{2}(a+b), \sqrt{a\,b}\right) \\ \operatorname{agm}\left(1, \sqrt{1-x^2}\right) &= \operatorname{agm}(1+x, 1-x) \\ \operatorname{agm}(1, b) &= \frac{1+b}{2}\operatorname{agm}\left(1, \frac{2\sqrt{b}}{1+b}\right).\end{aligned}$$

Solutions to the differential equation

$$(x^3 - x)\frac{d^2 y}{d x^2} + (3x^2 - 1)\frac{d y}{d x} + x\,y = 0$$

are given by $[\operatorname{agm}(1+x, 1-x)]^{-1}$ and $[\operatorname{agm}(1, x)]^{-1}$.

A generalization of the arithmetic-geometric mean is

$$I_p(a, b) = \int_0^\infty \frac{x^{p-2}\,d x}{(x^p + a^p)^{1/p}\,(x^p + b^p)^{(p-1)/p}},$$

which is related to solutions of the differential equation

$$x(1 - x^p)\,Y'' + [1 - (p+1)\,x^p]\,Y' - (p-1)\,x^{p-1}\,Y = 0.$$

The case $p = 2$ corresponds to the arithmetic-geometric mean via

$$I_2(a, b) = \int_0^\infty \frac{d x}{\sqrt{(x^2 + a^2)(x^2 + b^2)}} = \frac{\pi}{2\operatorname{agm}(a, b)}.$$

The case $p = 3$ gives the cubic relative

$$I_3(a, b) = \int_0^\infty \frac{x\,d x}{[(a^3 + x^3)(b^3 + x^3)^2]^{1/3}}$$

$$= \frac{\Gamma^3\left(\frac{1}{3}\right) {}_2F_1\left(\frac{1}{3}, \frac{1}{3}; \frac{2}{3}; (a/b)^3\right)}{2\pi b\sqrt{3}} - \frac{4a\pi^2 {}_2F_1\left(\frac{2}{3}, \frac{2}{3}; \frac{4}{3}; (a$$
$$\overline{3b^2\Gamma^3\left(\frac{1}{3}\right)}$$

discussed by Borwein and Borwein (1990, 1991) and Borwein (1996). For $a, b > 0$, this func
functional equation

$$I_3(a, b) = I_3\left(\frac{a+2b}{3}, \left[\frac{b}{3}(a^2 + ab + b^2)\right]^{1/3}\right).$$

It therefore turns out that for iteration with $a_0 = a$ and $b_0 = b$ and

$$a_{n+1} = \frac{a_n + 2b_n}{3}$$
$$b_{n+1} = \left[\frac{b_n}{3}(a_n^2 + a_n b_n + b_n^2)\right]^{1/3},$$

so

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n = \frac{I_3(1, 1)}{I_3(a, b)},$$

where

$$I_3(1, 1) = \frac{2\pi}{3\sqrt{3}}.$$

SEE ALSO: Arithmetic Mean, Arithmetic-Harmonic Mean, Gauss's Constant, Geometric Mean,
Function. [Pages Linking Here]

RELATED WOLFRAM SITES:
http://functions.wolfram.com/EllipticFunctions/ArithmeticGeometricMean/

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